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Symmetries of a (2+1)-dimensional breaking soliton equation

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Abstract. Infinitely many symmetries for a (2+1)-dimensional breaking soliton equation are constructed via the infinitesimal version of the 'dressing' method. These symmetries are proved to constitute an infinite-dimensional Lie algebra which contains some Abelian and Virasoro subalgebras. The hierarchies of equations generated by these symmetries are also considered: these hierarchies of equations are proved to be associated with the isospectral and non-isospectral deformations of the AKNS spectral problem.

1. Introduction

It is well known that symmetries play an important role in the study of integrable nonlinear evolution equations (INEE), such as the KdV equation and KP equation. It has been shown that almost all the well-known INEE possess infinitely many symmetries, and these symmetries usually constitute some infinite-dimensional Lie algebras [1–6]. This property of INEE is significant for better understanding of the integrability of these equations. The symmetries of an INEE also generate hierarchies of NEE which are associated with the isospectral and non-isospectral deformations of certain spectral problem, with the given INEE contained in the isospectral hierarchy of equations. These hierarchies of NEE are also integrable in the sense that they can be solved via the inverse scattering method.

In this paper, we shall consider the symmetries of the following (2+1)-dimensional NEE:

$$q_t = \mathrm{i}q_{xy} - 2\mathrm{i}q\partial_x^{-1}(qr)_y \tag{1.1a}$$

$$r_t = -\mathrm{i}r_{xy} + 2\mathrm{i}r\partial_x^{-1}(qr)_y. \tag{1.1b}$$

This equation is typical of the so-called 'breaking soliton' equations, which were studied by O I Bogoyovlenskii in a series of papers [7, 8]. Similar equations were also studied in [9]. These breaking soliton equations were used to describe the (2+1)-dimensional interaction of Riemann wave propagation along the y-axis with long-wave propagation along the x-axis. The simplest 'breaking soliton' solution of the equation (1.1), for example, can be put into the following form [7, 8]:

$$(q(x,y,t), r(x,y,t)) = (Q(x,t,\lambda(y,t)), R(x,t,\lambda(y,t))) \tag{1.2}$$

where for fixed value of $\lambda(y,t)$, (Q,R) gives a soliton solution for the nonlinear Schrödinger equation, while $\lambda(y,t)$ satisfies the Riemann wave equation. For any fixed initial value of $\lambda(y,t)$, $\lambda(y,t)$ ultimately 'breaks' to be a multi-valued function, thus (Q,R) also 'breaks' to give a multi-valued solution of the equation (1.1). [7, 8] presented the Lax pairs and the Hamiltonian structures for these equations, and showed that these equations can be solved via the inverse scattering method. [10] showed by using a recursion operator that equation (1.1) possesses an infinite set of symmetries, these symmetries constitute an infinite-dimensional Lie algebra.

In section 2 we shall construct symmetries of equation (1.1) by using the infinitesimal version of the 'dressing' method; we then show in section 3 that these symmetries constitute an infinite-dimensional Lie algebra, which contains some Abelian and Viarasoro subalgebras. In particular we show that the symmetry algebra constructed in [10] is only a subalgebra of the present algebra. In section 4 we shall consider the hierarchies of equations generated by these symmetries. These hierarchies of equations are proved to be associated with the isospectral and non-isospectral deformations of the well known AKNS spectral problem.

2. Symmetries of equation (1.1)

In this section, we shall construct the symmetries of (1.1) by using the infinitesimal version of the 'dressing' method [4]. Equation (1.1) has the following Lax pair:

$$\psi_x = M\psi \tag{2.1a}$$

$$\psi_t = 2\xi\psi_y + N\psi \tag{2.1b}$$

where

$$M=\begin{pmatrix} -\mathrm{i}\xi & q \\ r & \mathrm{i}\xi \end{pmatrix} \qquad N=\begin{pmatrix} -\mathrm{i}\partial_x^{-1}(qr)_y & \mathrm{i}q_y \\ -\mathrm{i}r_y & \mathrm{i}\partial_x^{-1}(qr)_y \end{pmatrix}$$

and

$$\xi_x = 0 \qquad \xi_t = 2\xi\xi_y.$$

Equation (1.1) is then equivalent to the following compatibility condition of equations (2.1a) and (2.1b):

$$M_t - N_x - [N, M] - 2\xi M_y = 0 \tag{2.2}$$

where $[A,B]=AB-BA$ for two operators A, B. Since it does not influence our construction of symmetries of (1.1), we assume for simplicity in this section that $\xi_y=0$. When $\xi_y\neq 0$, we can also apply the following approach to construct symmetries of (1.1), and the result is the same.

To construct symmetries of (1.1) by using the infinitesimal version of the 'dressing' method [4], we give the eigenfunction ψ an infinitesimal 'dressing'; $\psi\mapsto(1+\varepsilon\delta\chi)\psi$, where $\delta\chi$ is an appropriate operator, and ε is an infinitesimal parameter. Then, M and N also have an infinitesimal change: $M\mapsto M+\varepsilon\delta M$, $N\mapsto N+\varepsilon\delta N$. From equations (2,1$a$) and (2.1$b$) we have

$$\delta M = [\partial_x - M, \delta\chi] \tag{2.3a}$$

$$\delta N = 2[\delta\chi, \xi]\partial_y + [\delta\chi, N] + (\delta\chi)_t - 2\xi(\delta\chi)_y. \tag{2.3b}$$

A direct calculation shows that if $[(\delta\chi)_x, \xi]=0$, then $\delta M, \delta N$ defined by (2.3a) and

(2.3b) satisfy the following linearized equation of (2.2):

$$(\delta M)_t-(\delta N)_x-[\delta N,M]-[N,\delta M]-2\xi(\delta M)_y=0. \tag{2.4}$$

So, if we can find operator $\delta\chi$ such that δM and δN have the following form:

$$\delta M=\begin{pmatrix}0 & \delta q\\ \delta r & 0\end{pmatrix} \qquad \delta M=\begin{pmatrix}-\mathrm{i}\partial_x^{-1}(q\delta r+r\delta q)_y & \mathrm{i}(\delta q)_y\\ -\mathrm{i}(\delta r)_y & \mathrm{i}\partial_x^{-1}(q\delta r+r\delta q)_y\end{pmatrix} \tag{2.5}$$

where $\delta q,\delta r$ are scalar functions and independent of the spectral parameter ξ, then $(\delta q,\delta r)$ is a solution of the linearized equation of (1.1), thus $(\delta q,\delta r)$ is a symmetry of (1.1).

Motivated by the above, we now consider the following three types of $\delta\chi$:

$$\delta\chi_n=\xi^n\partial_y+\sum_{j=0}^{n}K_j\xi^{n-j} \qquad n\geqslant 0 \tag{2.6}$$

$$\delta\chi_n^k=\xi^{n-k}(\xi t+\tfrac{1}{2}y)^k\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}+\sum_{j=0}^{n}K_j\xi^{n-j} \qquad n\geqslant k+1 \qquad k\geqslant 0 \tag{2.7}$$

$$\delta\chi_{n,l}^{k+1}=(\xi t+\tfrac{1}{2}y)^k[-2t\xi^{n-k}+l\xi^{n-k-1}(\xi t+\tfrac{1}{2}y)]\partial_y+\xi^{n-k}(\xi t+\tfrac{1}{2}y)^k\partial_\xi+\sum_{j=0}^{n}K_j\xi^{n-j}$$

$$k\geqslant 0 \qquad n\geqslant k+1 \qquad l=1,2. \tag{2.8}$$

We note that if we substitute the above $\delta\chi$ into equations (2.3a) and (2.3b), then we find that the operator terms of the right-hand sides of these equations vanish. In what follows we always assume that we choose integral constants to be zero, and

$$K_j=\begin{pmatrix}A_j & C_j\\ B_j & -A_j\end{pmatrix}.$$

We first consider $\delta\chi_n$ given in (2.6). From the requirement that δM satisfies the condition (2.5), we find that equation (2.3a) is equivalent to the following relations:

$$K_0=0 \qquad K_1=\tfrac{1}{2}N \qquad C_1=\frac{\mathrm{i}}{2}q_y \qquad B_1=-\frac{\mathrm{i}}{2}r_y$$

$$\begin{pmatrix}-C_{j+1}\\ B_{j+1}\end{pmatrix}=\Phi\begin{pmatrix}-C_j\\ B_j\end{pmatrix} \qquad A_{j+1}=\partial_x^{-1}(qB_{j+1}-rC_{j+1}) \qquad j\geqslant 1 \tag{2.9}$$

$$\begin{pmatrix}\delta q\\ \delta r\end{pmatrix}=\Phi^n\begin{pmatrix}q_y\\ r_y\end{pmatrix}$$

where

$$\Phi=\frac{1}{2i}\begin{pmatrix}-\partial_x+2q\partial_x^{-1}r & 2q\partial_x^{-1}q\\ -2r\partial_x^{-1}r & \partial_x-2r\partial_x^{-1}q\end{pmatrix} \tag{2.10}$$

and K_j is uniquely determined by equation (2.3a) up to integral constants.

When $n=0$ it is easy to prove that $\delta\chi_0$ also satisfies equation (2.3b) with δN defined by (2.5), so $(\delta q,\delta r)=(q_y,r_y)$ is a symmetry of equation (1.1). For $n\geqslant 1$, instead of proving that the above $\delta\chi_n$ also satisfies equation (2.3b) with δN defined by (2.5) and δq, δr defined by (2.9), we adopt the following more convenient way to prove that $(\delta q, \delta r)$ is a symmetry of equation (1.1). It is not hard to prove that the operator Φ defined by (2.10) is a strong symmetry for equation (1.1), so from [11] we know that $(\delta q, \delta r)$ defined by (2.9) is asymmetry of equation (1.1). We denote this symmetry by τ_n.

Secondly, we consider $\delta\chi_n^k$ given in (2.7). Substitute this $\delta\chi$ into equation (2.3*a*); we know that K_j can be uniquely determined under our assumption of integral constants and by our requirement that δM satisfies the condition (2.5). In this way, from $\delta\chi_n^k$ we obtain a unique pair of $(\delta q, \delta r)$ which we denote by τ_n^k. For example, when $k=0$ we have

$$K_0=0 \qquad K_1=\begin{pmatrix}0 & q\\ r & 0\end{pmatrix} \qquad \begin{pmatrix}-C_{j+1}\\ B_{j+1}\end{pmatrix}=\Phi\begin{pmatrix}-C_j\\ B_j\end{pmatrix} \qquad j\geqslant 1$$
$$A_j=\partial_x^{-1}(qB_j-rC_j) \qquad j\geqslant 2 \qquad \begin{pmatrix}\delta q\\ \delta r\end{pmatrix}=\Phi^{n-1}\begin{pmatrix}q_x\\ r_x\end{pmatrix} \qquad n\geqslant 1. \tag{2.11}$$

Similar to the first case, we can prove that τ_n^0 is a symmetry of equation (1.1) for $n\geqslant 1$. We now claim in general that τ_n^k for $k\geqslant 1$, $n\geqslant k+1$ is also a symmetry of equation (1.1); this fact will be proved in the next section.

Third, we consider $\delta\chi_{n,l}^{k+1}$ given in (2.8). Substitute this $\delta\chi$ into equation (2.3*a*); we find that K_j can be uniquely determined under our assumption of integral constants and by our requiremnt that δM satisfies the condition (2.5). We denote the $(\delta q, \delta r)$ obtained from $\delta\chi_{n,l}^{k+1}$ by $\tau_{n,l}^{k+1}$. For example, when $k=0$, $l=2$ we have

$$K_0=\begin{pmatrix}\mathrm{i}x & 0\\ 0 & -\mathrm{i}x\end{pmatrix} \qquad K_1=\begin{pmatrix}0 & -xq\\ -xr & 0\end{pmatrix} \qquad \begin{pmatrix}-C_2\\ B_2\end{pmatrix}=\frac{1}{2\mathrm{i}}\begin{pmatrix}-(xq)_x+yq_y\\ -(xr)_x+yr_y\end{pmatrix}$$
$$\begin{pmatrix}-C_{j+1}\\ B_{j+1}\end{pmatrix}=\Phi\begin{pmatrix}-C_j\\ B_j\end{pmatrix} \qquad A_j=\partial_x^{-1}(qB_j-rC_j) \qquad j\geqslant 2 \tag{2.12}$$
$$\begin{pmatrix}\delta q\\ \delta r\end{pmatrix}=\Phi^{n-1}\begin{pmatrix}-(xq)_x+yq_y\\ -(xr)_x+yr_y\end{pmatrix} \qquad n\geqslant 1.$$

When $k=1$, $l=1$ we have

$$K_0=\begin{pmatrix}\mathrm{i}xt & 0\\ 0 & -\mathrm{i}xt\end{pmatrix} \qquad K_1=\frac{1}{2\mathrm{i}}\begin{pmatrix}xy-t^2\partial_x^{-1}(qr)_y & t^2q_y-2\mathrm{i}xtq\\ -t^2r_y-2\mathrm{i}xtr & xy+t^2\partial_x^{-1}(qr)_y\end{pmatrix}$$
$$\begin{pmatrix}-C_2\\ B_2\end{pmatrix}=\Phi\begin{pmatrix}-C_1\\ B_1\end{pmatrix}+\frac{1}{4}\begin{pmatrix}xyq\\ -xyr\end{pmatrix} \qquad \begin{pmatrix}-C_3\\ B_3\end{pmatrix}=\Phi\begin{pmatrix}-C_2\\ B_2\end{pmatrix}+\frac{1}{8\mathrm{i}}\begin{pmatrix}y^2q_y\\ y^2r_y\end{pmatrix}$$
$$A_2=\tfrac{1}{4}[2\mathrm{i}xtqr+2\mathrm{i}t\partial_x^{-1}(qr)+t^2\partial_x^{-1}(qr_{xy}-q_{xy}r)] \tag{2.13}$$
$$\begin{pmatrix}-C_{j+1}\\ B_{j+1}\end{pmatrix}=\Phi\begin{pmatrix}-C_j\\ B_j\end{pmatrix} \qquad A_j=\partial_x^{-1}(qB_j-rC_j) \qquad j\geqslant 3$$
$$\begin{pmatrix}\delta q\\ \delta r\end{pmatrix}=\Phi^{n-2}\begin{pmatrix}-2\mathrm{i}C_3\\ 2\mathrm{i}B_3\end{pmatrix} \qquad n\geqslant 2.$$

Then, similar to the second case, we can prove that $\tau_{2,l}^2, \tau_{n,l}^1$ are symmetries of equation (1.1) for $n\geqslant 1$, $l=1,2$, and we claim in general that $\tau_{n,l}^k$ is also a symmetry for $k \geqslant 1$, $n \geqslant k+1$, $l=1,2$, this fact will also be proved in the next section.

So, in this section we have constructed infinitely many symmetries $\tau_n (n \geqslant 0)$, $\tau_n^k (n \geqslant k+1, k \geqslant 0)$, $\tau_{n,l}^{k+1} (k \geqslant 0, n \geqslant k+1, l=1,2)$ for equation (1.1). We note that the symmetries constructed in [9] correspond to τ_n, τ_n^0, $\tau_{n,l}^1 (l=1,2)$. In the next secton we shall consider the algebraic structure of these symmetries.

3. Algebraic structure of the symmetries of equation (1.1)

In this section, we shall consider the algebraic structure of the symmetries of equation (1.1) constructed in section 2. For two symmetries of equation (1.1) τ and σ, we define their commutator as follows:

$$\{\tau, \sigma\} = \tau'[\sigma] - \sigma'[\tau] \tag{3.1}$$

where

$$\tau'[\sigma] = \lim_{\varepsilon \to 0} \frac{\partial}{\partial \varepsilon} \tau(\omega + \varepsilon \sigma) \qquad \omega = (q, r)$$

is the Gateaux derivative of τ in the direction of σ. Then it is well known that $\{\tau, \sigma\}$ is also a symmetry of equation (1.1), and $\{,\}$ is a Lie bracket.

We first calculate $\{\tau_n, \tau_m\}$ for $n, m \geqslant 0$. We denote δM corresponding to the $\delta\chi_n$ by δM_n. Then by our definition we have

$$\begin{aligned}
\{\delta M_n, \delta M_m\} &= (\delta M_n)'[\delta M_m] - (\delta M_m)'[\delta M_n] \\
&= [-M'[\delta M_m], \delta\chi_n] + [\partial_x - M, (\delta\chi_n)'[\delta M_m]] \\
&\quad - [-M'[\delta M_n], \delta\chi_m] - [\partial_x - M, (\delta\chi_m)'[\delta M_n]] \\
&= [\partial_x - M, (\delta\chi_n)'[\delta M_m] - (\delta\chi_m)'[\delta M_n] + [\delta\chi_n, \delta\chi_m]].
\end{aligned}$$

The operator $(\delta\chi_n)'[\delta M_m] - (\delta\chi_m)'[\delta M_n] + [\delta\chi_n, \delta\chi_m]$ has the form $\Sigma_{j=0}^{n+m-1} K_j$ and it is easy to see that the leading terms K_0, K_1 are zero. This operator satisfies equation (2.3*a*) with δM defined by $\{\delta M_n, \delta M_m\}$. Since equation (2.3*a*) uniquely determines K_j up to integral constants, we know from our assumption on integral constants that this operator just equals zero. Thus we have $\{\delta M_n, \delta M_m\} = 0$, which we also write as

$$\{\tau_n, \tau_m\} = 0. \tag{3.2}$$

Since τ_n^0 and $\tau_{2,1}^2$ are symmetries of equation (1.1), similar to the above calculation we have

$$\{\tau_n^0, \tau_{2,1}^2\} = -n\tau_{n+1}^1. \tag{3.3}$$

Thus we have proved that τ_n^1 is also a symmetry of equation (1.1) for $n \geqslant 2$. We then calculate $\{\tau_n^1, \tau_{2,1}^2\}$, which turns out to be $(\frac{1}{2} - n)\tau_{n+1}^2$. So τ_n^2 is a symmetry of equation (1.1) for $n \geqslant 3$. In this way, we can prove step by step that τ_n^k is a symmetry of equation (1.1) for any $k \geqslant 0$, $n \geqslant k+1$. Similar to the above argument, we can prove that $\tau_{n,l}^{k+1}$ is a symmetry for any $k \geqslant 0$, $n \geqslant k+1$ and $l=1,2$, this fact can also be seen more clearly from the following commutation relations of the symmetries of equation (1.1):

$$\{\tau_n, \tau_j^0\} = 0 \qquad \{\tau_n, \tau_m^l\} = \frac{l}{2}\tau_{n+m-1}^{l-1} \qquad n \geqslant 0 \qquad j, l \geqslant 1 \qquad m \geqslant l+1 \tag{3.4}$$

$$\{\tau_n, \tau^1_{j,1}\} = (\tfrac{1}{2} - n)\tau_{n+j-1} \qquad \{\tau_n, \tau^1_{j,2}\} = (1-n)\tau_{n+j-1} \qquad n \geqslant 0 \qquad j \geqslant 1 \tag{3.5}$$

$$\{\tau_n, \tau^{l+1}_{m,1}\} = \left(n - \frac{1-l}{2}\right)\tau^l_{n+m-1,1} + (\tfrac{1}{2} - n)\tau^l_{n+m-1,2} \tag{3.6}$$

$$\{\tau_n, \tau^{l+1}_{m,2}\} = (n-1)\tau^l_{n+m-1,1} + \left(\frac{l}{2} - n + 1\right)\tau^l_{n+m-1,2} \qquad n \geqslant 0 \qquad l \geqslant 1 \qquad m \geqslant l+1 \tag{3.7}$$

$$\{\tau^k_n, \tau^l_m\} = 0 \qquad \{\tau^k_n, \tau^{l+1}_{m,2}\} = -n\tau^{k+1}_{n+m-1} \tag{3.8}$$

$$\{\tau^k_n, \tau^{l+1}_{m,1}\} = \left(\frac{k}{2} - n\right)\tau^{k+l}_{n+m-1} \tag{3.9}$$

$$\{\tau^{k+1}_{n,1}\, \tau^{l+1}_{m,1}\} = \left(m - n + \frac{k-l}{2}\right)\tau^{k+l+1}_{n+m-1,1} \tag{3.10}$$

$$\{\tau^{k+1}_{n,1}\, \tau^{l+1}_{m,2}\} = (1-n)\tau^{k+l+1}_{n+m-1,1} + \left(m - 1 - \frac{l}{2}\right)\tau^{k+l+1}_{n+m-1,2} \tag{3.11}$$

$$\{\tau^{k+1}_{n,2}\, \tau^{l+1}_{m,2}\} = (m-n)\tau^{k+l+1}_{n+m-1,2} \qquad k, l \geqslant 0 \qquad n \geqslant k+1 \qquad m \geqslant l+1. \tag{3.12}$$

The derivation of the commutation relations (3.4)–(3.12) is similar to that of the commutation relation (3.2), so we omit it here.

From relations (3.2)–(3.12) we see that the symmetries of equation (1.1) constructed in section 2 form a basis of an infinite-dimensional Lie algebra under the Lie bracket $\{,\}$, and this Lie algebra contains some Abelian subalgebras and Virasoro subalgebras (without centre). This property is commonly shared by almost all the well known $(1+1)$-dimensional and $(2+1)$-dimensional soliton equations.

Remark 1. We see from relations (3.2)–(3.12) that the above infinite-dimensional Lie algebra is generated by the symmetries τ_n, τ^0_m, $\tau^1_{m,l}$, $\tau^2_{2,1}$ ($n \geqslant 0$, $m \geqslant 1$, $l = 1, 2$).

Remark 2. From [2, 4] we know that the symmetries of the $(2+1)$-dimensional KP equation are indexed by two integers, so it seems to us that the symmetry algebra of equation (1.1) is bigger than that of the KP equation, this may also suggests some differences between the usual $(2+1)$-dimensional soliton equations and the $(2+1)$-dimensional braking soliton equation (1.1).

4. Isospectral and non-isospectral hierarchies of equations

In this section we consider the hierarchies of equations generated by the symmetries of equation (1.1) constructed in section 2. We denote δq and δr corresponding to the symmetry τ_n by δq_n and δr_n, respectively; similarly we define δq^k_n, δr^k_n, $\delta q^{k+1}_{n,l}$, $\delta r^{k+1}_{n,l}$. Then the hierarchies of equations are given as follows:

$$\begin{pmatrix} q_{t_n} \\ r_{t_n} \end{pmatrix} = \begin{pmatrix} \delta q_n \\ \delta r_n \end{pmatrix} = \Phi^n \begin{pmatrix} q_y \\ r_y \end{pmatrix} \qquad n \geqslant 0 \tag{4.1}$$

$$\begin{pmatrix} q_{t^k_n} \\ r_{t^k_n} \end{pmatrix} = \begin{pmatrix} \delta q^k_n \\ \delta r^k_n \end{pmatrix} \qquad k \geqslant 0, \qquad n \geqslant k+1 \tag{4.2}$$

$$\begin{pmatrix} q_{t^{k+1}_{n,l}} \\ r_{t^{k+1}_{n,l}} \end{pmatrix} = \begin{pmatrix} \delta q^{k+1}_{n,l} \\ \delta r^{k+1}_{n,l} \end{pmatrix} \qquad k \geqslant 0, \qquad n \geqslant k+1. \tag{4.3}$$

When $n=1$, equation (4.2) is equivalent to equation (1.1), and the flows given by equations (4.1) and (4.2) are mutually commutative. When $k=0$, the hierarchy of equations given in (4.2) is just the usual KdV hierarchy. It is easy to see from equation (2.3*a*) and conditon (2.5) that equation (4.1) has the following Lax pair:

$$\psi_x = M\psi \qquad \psi_{t_n} = \delta\chi_n \psi \tag{4.4}$$

where M is given in (2.1), ξ satisfies

$$\xi_x = 0 \qquad \xi_{t_n} = \xi^n \xi_y \tag{4.5}$$

and $\delta\chi_n$ is given by (2.6). Equation (4.2) has the following Lax pair:

$$\psi_x = M\psi \qquad \psi_{t^k_n} = \delta\chi^k_n \psi \tag{4.6}$$

where

$$\xi_x = 0 \qquad \xi_{t^k_n} = 0 \tag{4.7}$$

and $\delta\chi^k_n$ is given by (2.7). The Lax pair of equation (4.3) is given by

$$\psi_x = M\psi \tag{4.8a}$$

$$\psi_{t^{k+1}_{n,l}} = \left\{ (\xi t + \tfrac{1}{2}y)^k [-2t\xi^{n-k} + l\xi^{n-k-1}(\xi t + \tfrac{1}{2}y)]\partial_y + \sum_{j=0}^{n} K_j \xi^{n-j} \right\} \psi \tag{4.8b}$$

where K_j is given in (2.8), and ξ satisfies

$$\xi_x = 0 \qquad \xi_{t^{k+1}_{n,l}} = (\xi t + \tfrac{1}{2}y)^k [-2t\xi^{n-k} + l\xi^{n-k-1}(\xi t + \tfrac{1}{2}y)]\xi_y - \xi^{n-k}(\xi t + \tfrac{1}{2}y)^k. \tag{4.9}$$

The Lax pairs (4.6) and (4.8) are also derived from equation (2.3*a*) and condition (2.5).

From the above Lax representations of the equations (4.1)–(4.3), we see that the hierarchy of equations given in (4.2) corresponds to the isospectral deformation of the spectral problem $\psi_x = M\psi$, which is the well known AKNS spectral problem, whereas the hierarchies of equations given in (4.1) and (4.3) correspond to the non-isospectral deformation of the same spectral problem. So these equations can be solved by using the inverse scattering method on the AKNS spectral problem; the evolution of the scattering data can be evaluated from equations (4.4), (4.6) and (4.8*b*).

5. Conclusion

We have constructed infinitely many symmetries for the $(2+1)$-dimensional breaking soliton equation (1.1), these symmetries are proved to constitute an infinite-dimensional Lie algebra with some Abelian and Virasoro subalgebras. This symmetry algebra seems to be bigger than that of the usual $(2+1)$-dimensional soliton equations, such as the KP equation. This fact may help us to understand the differences between the usual $(2+1)$-dimensional soliton equations and the $(2+1)$-dimensional breaking soliton equations. We have also considered the hierarchies of equations generated by these symmetries; they correspond to the isospectral and non-isospectral deformations of the well-known AKNS spectral problem. Similar results can also be obtained for other $(2+1)$-dimensional breaking soliton equations given in [7, 8], such as the following $(2+1)$-dimensional breaking soliton equation:

$$u_{xt} = 4u_x u_{xy} + 2u_y u_{xx} - u_{xxxy}$$

details will be given elsewhere.

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